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On the Stability of Perturbations of Optimal Control Problems

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1. INTRODUCTION

Cullum [1, 2] considered the perturbations of optimal control problems. Let P be an optimal control problem describable by a system of ordinary differential equations and with an integral cost functional. Perturb the system of differential equations associated with P, the boundary conditions, and the control sets in such a way that the alterations are describable by a single parameter ε. Denote the corresponding problem gererated from P by $P(\varepsilon)$. If optimal solutions exist for $P(\varepsilon)$, what is the relationship between these and the optimal solutions of the original problem $P = P(0)$?

Kirillova [3] considered this question for a particular linear, time-optimal control problem. Kirilova proved that, given $\varepsilon_n \to 0$, the optimal times for the corresponding problems $P(\varepsilon_n)$ converge to the optimal times for $P(0)$, and the corresponding sequence of optimal controls converges in measure to the optimal control for P. Cullum's papers extend the results to the above problems. If $\varepsilon_n \to 0$ and $(x(n), u(n), l_n, a_n, b_n)$ is any sequence of admissible trajectories for the corresponding problems $P(\varepsilon_n)$, then there exists (x_0, u_0, l_0, a, b) such that x_0 is an admissible trajectory for $P(0)$ with control u_0, and $l_n \to l_0$, $(a_n, b_n) \to (a_0, b_0)$, $x_n(t)$ converges to $x_0(t)$ for each $t \in l_0$, and there exist measurable extensions $\bar{u}_n$ of u_n to l_0 such that $\bar{u}_n$ converges to u_0 in the weak topology of $L_2(l_0)$. If P has a unique optimal solution (x_0, u_0), then there exist similar results for minimizing sequences.

For the relationship between optimal solutions of $P(\varepsilon)$ and optimal solutions of $P(0)$ we can put forward the problem in another way. If u_0 is an optimal solution of $P(0)$, do any optimal solutions of $P(\varepsilon)$ exist in each neighborhood of u_0?In ther words, given $\varepsilon_n \to 0$, does any sequence $(u_n)_{n \in N}$ of optimal solutions of $P(\varepsilon_n)$ exist such that u_n converges to u_0?

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2. Statement of the Problem

We consider the control systems

$$\begin{aligned} \dot{x}(t) &= \hat{f}(\varepsilon, x(t), u(t), t) \qquad \text{a.e.}, \\ x(0) &= x_0. \end{aligned} \tag{2.1}$$

$x(t) \in E^n$ denotes the state, $u(t) \in E^m$ the control, where E^n, E^m are n-dimensional Euclidean space and m-dimensional Euclidean space, respectively. $l = [0, T]$ is a fixed interval in E^1. For each $\varepsilon \in [0, 1]$, $U(\varepsilon) \subset E^m$, u: $[0, T] \to E^m$, are measurable vector functions such that $u(t) \in U(\varepsilon)$. u is called an admissible control function of control system (2.1). The corresponding solutions (x, l) of (2.1) are called admissible trajectories. For a fixed $\varepsilon \in [0, T]$, the optimal control problem $P(\varepsilon)$ is to give an admissible control u and corresponding admissible trajectory x to minimize the following cost functional

$$\int_0^T f_0(\varepsilon, x(t), u(t), t)\, dt. \tag{2.2}$$

Several symbols are introduced as follows:

$$f(\varepsilon, x, u, t) = (f_0(\varepsilon, x, u, t), \hat{f}(\varepsilon, x, u, t)); \tag{2.3}$$

for any number $\alpha > 0$,

$$U(U(0), \alpha) = \{v \in E^m \mid |u - v| < \alpha, u \in U(0)\}, \tag{2.4}$$

$$S_0 = \{u \in L_2(l) \mid u(t) \in U(0), \text{a.e.}\}, \tag{2.5}$$

$$S_\varepsilon = \{u \in L_2(l) \mid u(t) \in U(\varepsilon), \text{a.e.}\}, \tag{2.6}$$

$$(S_0, \alpha) = \{u \in L_2(l) \mid u(t) \in U(U(0), \alpha), \text{a.e.}\}. \tag{2.7}$$

The global assumptions are:

(a) Let $\{U(\varepsilon) \subset E^m\}$ be a family of compact convex subsets of E^m, which multivalued mapping from $[0, T]$ into E^m is both upper semicontinuous and lower semicontinuous at $\varepsilon = 0$;

(b) For each $\varepsilon \in [0, T]$, $f(\varepsilon, x, u, t)$ is continuous on $E^n \times U(U(0), \alpha)$ for each t and measurable in t for each (x, u), where $(x, u, t) \in [0, l] \times E^n \times U(U(0), \alpha)$. For all $(\varepsilon, x, u) \in [0, 1] \times E^n \times U(U(0), \alpha)$,

$$|f(\varepsilon, x, u, t)| \leqslant \mu(t), \qquad \text{a.e.}, \tag{2.8}$$

where $\mu(t) \in L_1(l)$. Given $\varepsilon \in [0, l]$ for all $u \in U(U(0), \alpha)$ and $x', x'' \in E^n$,

$$|\hat{f}(\varepsilon, x', u, t) - \hat{f}(\varepsilon, x'', u, t)| \leqslant \lambda(t)\, |x' - x''|, \qquad \text{a.e.}, \tag{2.9}$$

where $\lambda(t) \in L_1(l)$. For all $u \in U(U(0), \alpha)$, and $x', x'' \in E^n$,

$$|f_0(0, x', u, t) - f_0(0, x'', u, t)| \leqslant \gamma(t)\,|x' - x''|, \qquad \text{a.e.,} \tag{2.10}$$

where $\gamma(t) \in L_1(l)$.

Remark 2.1. The inequality (2.10) of condition (b) can be relaxed as follows. For each $t \in [0, T]$, $\mathscr{F}_t = \{f_0(0, x, u, t) \mid u \in U(U(0), \alpha)\}$ is a family of functions of x. $\mathscr{F}_t$ is equicontinuous on E^n.

(c) For almost every $t \in l$, $f_0(\varepsilon, x, u, t)$ converges to $f_0(0, x, u, t)$ uniformly on $E^n \times U(U(0), \alpha)$. $\hat{f}(\varepsilon, x, u, t)$ converges to $\hat{f}(0, x, u, t)$ uniformly on $E^n \times U(U(0), \alpha) \times l$.

The differential equations (2.1) are a Caratheodory system, which has a unique solution on l.

By Lemma 7 [1] and condition (a), S_0, S_ε are weakly closed convex subsets of $L_2(l)$.

3. Limiting Property of Admissible Control Set

In this section we consider a limiting property of S_ε at $\varepsilon = 0$. First, several definitions will be made.

Let X, Y be metric spaces, ρ metric functions, and M a multivalued mapping from X into Y.

Definition 1. The mapping M is said to be upper semicontinuous at x_0 if and only if

$$\begin{pmatrix} \forall\, \delta \in R^+ \\ \forall\, (x_n \in X)_N \to x_0 \end{pmatrix} (\exists\, n_0 \in N)(\forall\, n \geqslant n_0) \Rightarrow M(x_n) \subset U(M(x_0), \delta).$$

The mapping M is said to be lower semicontinuous at x_0 if and only if

$$\begin{pmatrix} \forall\, \delta \in R^+ \\ \forall (x_n \in X)_N \to x_0 \end{pmatrix} (\exists\, n_0 \in N)(\forall n \geqslant n_0) \Rightarrow M(x_0) \subset U(M(x_n), \delta).$$

Definition 2. The mapping M is said to be closed if and only if

$$\begin{pmatrix} \forall (x_n \in X)_N \to x_0 \\ \forall (y_n \in M(x_n))_N \to y_0 \end{pmatrix} \Rightarrow y_0 \in M(x_0).$$

The mapping M is said to be open if and only if

$$\begin{pmatrix} \forall (x_n \in X)_N \to x_0 \\ \forall y_0 \in M(x_0) \end{pmatrix} (\exists (y_n \in M(x_n))_N) \Rightarrow y_n \to y_0.$$

LEMMA 1. *If the family of sets* $\{U(\varepsilon) | \varepsilon \in [0, 1]\}$ *satisfies condition* (a), *then, given* $\varepsilon \to 0$, *the corresponding family* $\{S_\varepsilon\}$ *is both closed and opened at* $\varepsilon = 0$.

Proof. By Lemma 1 of [2], given $\varepsilon_n \to 0$, $(u_n \in S\varepsilon_n)_N$ converges weakly to u_0, then $u_0 \in S_0$. Obviously, if $(u_n \in S_{\varepsilon_n})_N$ converges strongly to u_0, we have still $u_0 \in S_0$. Hence $\{S_\varepsilon\}$ is a closed at $\varepsilon = 0$. (Henceforth we denote S_{ε_n} by S_n.)

For any $\varepsilon_n \to 0$ and any $\delta > 0$, by condition (a) there exists $n_0 \in N$ such that if $n \geqslant n_0$ we have

$$U(U(\varepsilon_n), \qquad \delta/\sqrt{T}) \supset U(0) \tag{3.1}$$

We denote

$$(S_n, \delta/\sqrt{T}) = \{u \in L_2[l] | u(t) \in U(U(\varepsilon_n), \delta/\sqrt{T}), \text{ a.e.}\}, \tag{3.2}$$

$$U(S_n, \delta) = \{v \in L_2[l] | \|v - u\| < \delta, u \in S_n\}. \tag{3.3}$$

Obviously, $(S_n, \delta/\sqrt{T}) \supset S_0 \forall n \geqslant n_0$. We find the following inference: For $n \geqslant n_0$,

$$U(S_n, \delta) \supset (S_n, \delta/\sqrt{T}) \supset S_0. \tag{3.4}$$

For any $u \in (S_n, \delta/\sqrt{T})$ we have

$$u(t) \in U(U(\varepsilon_n), \delta/\sqrt{T}) \text{ a.e..}$$

Given $n \geqslant n_0$, we now set

$$l_1 = \{t \in l | u(t) \in U(U(\varepsilon_n), \delta/\sqrt{T})\},$$
$$l_1^1 = \{t \in l_1 | u(t) \in U(\varepsilon_n)\},$$
$$l_1^2 = \{t \in l_1 | u(t) \in U(\varepsilon_n)\}.$$

We construct the function $\bar{u}_n$

$$\begin{aligned} \bar{u}_n(t) &= u(t), \qquad t \in l_1^1, \\ &= d_n(t), \qquad t \in l_1^2, \end{aligned}$$

where $|d_n(t) - u(t)| = \min_{v(t) \in U(\varepsilon_n)} |v(t) - u(t)|$, $t \in l_1^2$. The existence of $d_n(t)$ is trivial. We are going to prove the measurability of $d_n(t)$. Because $U(\varepsilon_n)$ is the bounded closed set of E^m and $u(t)$ is measurable, then l_1^1 is a measurable set of E^1. Hence $l_1^2 = l_1 - l_1^1$ is still a measurable set, and the function $u(t)$ can be approximated by a sequence of simple functions. The projection of a simple function on $U(\varepsilon_n)$ is still a simple function. Therefore, $d_n(t)$ is approximated by the sequence of the simple functions and $d_n(t)$ is the measurable function on l_1^2. As result we have $\bar{u}_n \in S_n$. It suffices that we prove that $y \in U(S_n, \delta)$

$$\|\bar{u}_n - u\| = \left(\int_{l_1^1} |\bar{u}_n(t) - u(t)|^2\, dt + \int_{l_1^2} |d_n(t) - u(t)|^2\, dt\right)^{1/2}$$
$$= \left(\int_{l_1^2} |d_n(t) - u(t)|^2\, dt\right)^{1/2} < \left(\int_{l_1^2} (\delta/\sqrt{T})^2\, dt\right)^{1/2} = \delta;$$

that is, $u \in U(S_n, \delta)$ and $U(S_n, \delta) \supset (S_n, \delta/\sqrt{T})$.

To sum up, for any $\delta > 0$, there exists $n_0 \in N$ such that for $n \geqslant n_0$,

$$U(S_n, \delta) \supset S_0.$$

Hence for a fixed $u_0 \in S_0$ there exist $u_n \in S_n$ such that $\|u_n - u_0\| < \delta$. Obviously, there exists a sequence $(u_n \in S_n)_N$ and $u_n \to u_0$; that is, the mapping S_ε is open at $\varepsilon = 0$.

4. The Stability of Perturbation

We denote the solutions of Eq. (2.1) by $x(\varepsilon, u, t)$. Then the cost functional (2.2) is a functional of the control u,

$$J(\varepsilon, u) = \int_0^T f_0(\varepsilon. x(\varepsilon, u, t), u(t), t)\, dt. \tag{4.1}$$

Hence, the optimal control problems $P(0)$, $P(\varepsilon)$, and following optimization problems are equivalent, respectively,

$$P(0): \min_{u \in S_0} J(0, u),$$

$$P(\varepsilon): \min_{u \in S_n} J(\varepsilon, u).$$

Definition 3. $\bar{u}_0$, $\bar{u}_\varepsilon$ are global minimums of $P(0)$, $P(\varepsilon)$, respectively, if $u_0 \in S_0$, $\bar{u}_\varepsilon \in S_\varepsilon$, and

$$J(0, \bar{u}_0) \leqslant J(0, u), \qquad \forall u \in S_0, \tag{4.2}$$

$$J(\varepsilon, \bar{u}_\varepsilon) \leqslant J(\varepsilon, u), \qquad \forall u \in S_\varepsilon. \tag{4.3}$$

DEFINITION 4. $\bar{u}_0, \bar{u}_\varepsilon$, are local minimums of $P(0)$, $P(\varepsilon)$, respectively, if $\bar{u}_0 \in S_0$, $\bar{u}_\varepsilon \in S_\varepsilon$, and there exist open neighborhoods G_0, G_ε, such that

$$J(0, u_0) \leqslant J(\varepsilon, u), \qquad \forall u \in S_0 \cap G_0, \tag{4.4}$$

$$J(\varepsilon, \bar{u}_\varepsilon) \leqslant J(\varepsilon, u), \qquad \forall u \in S_\varepsilon \cap G_\varepsilon. \tag{4.5}$$

LEMMA 2. *Assume that S_0 contains at least two points u_0, u_1 such that $\|u_0 - u_1\| = d > 0$, where d is any positive number. For any ω, $0 < \omega < d/2$, given $\varepsilon_n \to 0$, there exists $n_0 \in N$ such that for $n \geqslant n_0$,*

$$S_0' = \{u \mid u \in S_0, \|u - u_0\| = \omega\} \neq \varnothing,$$

$$S_n' = \{u \mid u \in S_n, \|u - u_0\| = \omega\} \neq \varnothing.$$

Proof. Given $0 < \omega < d/2$, $u_1 \bar{\in} B(u_0, \omega) = \{u \in L_2(I) \mid \|u - u_0\| \leqslant \omega\}$. Because S_0 is a convex set, the line segment $\overline{u_0 u_1}$ is contained in S_0, and $\overline{u_0 u_1}$ intersects the boundary curve of $B(u_0, \omega)$.

By Lemma 1, mapping S_ε is open at $\varepsilon = 0$. There exist two sequences $(u_n \in S_n)_N$, $(v_n \in S_n)_N$ and $n_0 \in N$ such that for $n \geqslant n_0$, $\|u_n - u_0\| < \omega$, $\|v_n - u_1\| < \omega$. Because S_n are convex sets, for $n \geqslant n_0$ the line segments $\overline{v_n u_n}$ are contained in S_n. Because $B(u_0, \omega) \cap B(u_1, \omega) = \phi$, all line segments $\overline{v_n u_n}$ intersect the boundary curve of $B(u_0, \varepsilon)$.

Now we introduce a variation of the dominated convergence theorem.

LEMMA 3. *Let $f_n(u(t), t)$, $f(u(t), t)$ be measurable functions on $[0, T]$, where $u(t) \in E^m$. We set $S = \{u \in L_2([0, T]) \mid u(t) \in U$, a.e.$\}$, where U is a subset of E^m. Assume that $f_n(u(t), t) \to f(u(t), t)$, a.e. and for each fixed t the convergence is uniform on U. Assume that there exists a function $u(t) \in L_1([0, T])$ such that for all $u(t) \in U$,*

$$|f_n(u(t), t)| \leqslant u(t), \qquad \text{a.e..}$$

Then $\int_0^T f_n(u(t), t)\, dt \to \int_0^T f(u(t), t)\, dt$, and the convergence is uniform on $u \in S$.

Proof. For each fixed $u \in S$, by the dominated convergence theorem we have

$$\int_0^T f_n(u(t), t)\, dt \to \int_0^T f(u(t), t)\, dt.$$

We want to prove that the convergence is uniform on $u \in S$. Assume that the conclusion is not true. Then for any $\delta > 0$ and each $k \in N$, there exist $n \geqslant k$ and functions $u_{n_k} \in S$ such that

$$\left| \int_0^T (f_{n_k}(t), t) - f(u_{n_k}(t), t))\, dt \right| > \delta. \tag{4.6}$$

We now set

$$g_{n_k}(t) = f_{n_k}(u_{n_k}(t), t) - f(u_{n_k}(t), t).$$

By the uniform convergence of $f_n(u(t), t)$ on $u(t) \in U$ for $k \to \infty$ we have $g_{n_k}(t) \to 0$ a.e., and $| g_{n_k}(t)| \leqslant 2\mu(t)$. By the dominated convergence theorem,

$$\int_0^T g_{n_k}(t)\, dt \to 0.$$

This contradicts (4.6) and the proof is complete.

LEMMA 4. *Assume that the conditions* (a)–(c) *are satisfied. Let* $x(n, u, t)$, $x(0, u, t)$ *be the solutions of* (2.1) *at* $\varepsilon = 0$, $\varepsilon = \varepsilon_n$, *respectively. Given* $\varepsilon_n \to 0$, *for almost* $t \in [0, T]$, $x(n, u, t)$ *converges to* $x(0, u, t)$ *uniformly on* (S_0, α).

Proof. For any (S_0, α) we have

$$\begin{aligned} &|x(n, u, t) - x(0, u, t)| \\ &\quad \leqslant \int_0^t |\hat{f}(\varepsilon_n, x(n, u, \tau), u(\tau), \tau) - \hat{f}(0, x(0, u, \tau), u(\tau), \tau)|\, d\tau \\ &\quad \leqslant \int_0^T |\hat{f}(\varepsilon_n, x(n, u, \tau), u(\tau), \tau) - \hat{f}(0, x(n, u, \tau), v(\tau), \tau)|\, d\tau \\ &\qquad + \int_0^T |\hat{f}(0, x(n, u, \tau), u(\tau), \tau) - \hat{f}(0, x(0, u, \tau), u(\tau), \tau)|\, d\tau. \end{aligned}$$

By condition (c), for any $\theta_t > 0$, there exists n_0 such that for $n \geqslant n_0$,

$$|\hat{f}(\varepsilon_n, x(n, u, \tau), u(\tau), \tau) - \hat{f}(0, x(n, u, \tau), u(\tau), \tau)| < \theta_t/T \cdot \exp \int_0^T \lambda(t)\, dt.$$

Then

$$\begin{aligned} &|x(n, u, t) - x(0, u, t)| \\ &\quad < \int_0^t \theta_t/T \exp \int_0^T \lambda(t)\, dt\, d\tau + \int_0^t \lambda(\tau)|x(n, u, \tau) - x(0, u, \tau)|\, d\tau. \end{aligned}$$

By Gronwell's lemma [5],

$$|x(n, u, t) - x(0, u, t)|$$
$$< \int_0^t \exp\left(\int_\tau^t \lambda(\eta)\, d\eta\right) \cdot \theta_t/T \exp \int_0^T \lambda(n)\, dn\, d\tau < \theta_t, \qquad \text{a.e..}$$

Because $u \in (S_0, \alpha)$ is arbitrary, for almost all $t \in [0, T]$, $x(n, u, t)$ converges to $x(0, u, t)$ uniformly on (S_0, α).

LEMMA 5. *Assume that the conditions* (a)–(c) *are satisfied. Given* $\varepsilon_n \to 0$, $J(\varepsilon_n, u)$ *converges to* $J(0, u)$ *uniformly on* (S_0, α).

Proof. In condition (b), $\gamma(t)$ is integrable and finite for almost all $t \in [0, T]$. Given $t \in [0, T]$ (except a set with measure zero), for any $\theta_t > 0$ there exists $n' \in N$ such that for $n \geqslant n'$,

$$|x(n, u, t) - x(0, u, t)| < \theta_t, \qquad \forall u \in (S_0, \alpha).$$

By condition (c), there exists $n_0 \in N$ such that for $n \geqslant n_0 \geqslant n'$,

$$|f_0(\varepsilon_n, x(n, u, t), u(t), t) - f_0(0, x(n, u, t), u(t), t)| < \theta_t,$$
$$\forall u(t) \in U(U(0), t).$$

Hence for $n \geqslant n_0$,

$$|f_0(\varepsilon_n, x(n, u, t), u(t), t) - f_0(0, x(0, u, t), \dot{u}(t), t)|$$
$$\leqslant |f_0(\varepsilon_n, x(n, u, t), u(t), t) - f_0(0, x(n, u, t), u(t), t)|$$
$$+ |f_0(0, x(n, u, t), u(t), t) - f_0(0, x(0, x(0, u, t), u(t), t)|$$
$$\leqslant \theta_t + \gamma(t)|x(n, u, t) - x(0, u, t)| < (1 + \gamma(t))\, \theta_t, \forall u(t) \in U(U(0), \alpha).$$

Since $\gamma(t)$ is finite for fixed t, for almost every t in $[0, T]$, $f_0(\varepsilon_n, x(n, u, t), u(t), t)$ converges to $f_0(0, u, t), u(t), t)$ uniformly on $u(t) \in U(U(0), t)$. By Lemma 3 the functional $J(\varepsilon_n, u)$ converges to $J(0, u)$ uniformly on (S_0, α).

THEOREM 1. *Let* $P(\varepsilon)(0 \leqslant \varepsilon \leqslant 1)$ *be a family of the optimal control problems that satisfy the global assumptions* (a)–(c), *and the following assumptions*:

(I). *There exist at least two measurable functions* u_1, u_0 *such that* $u_1(t), u_0(t) \in U(0)$, *a.e. and* $\|u_1 - u_0\| = \alpha > 0$, *where* α *is a constant.*

(II). *Let* u_0 *be a local minimum of* $P(0)$. *There exists a strict increasing real-valued function* $\tau(\xi)(\tau(0) = 0)$ *on* $[0, \infty)$ *such that for a sufficiently small* $\Delta > 0$,

$$J(0, u) - J(0, u_0) \geqslant \tau(\|u - u_0\|), \qquad \forall u \in B(u_0, \Delta) \cap (S_0, \alpha). \qquad (4.7)$$

(III). *For each* $\varepsilon \in [0, 1]$, $J(\varepsilon, u)$ *is weakly lower semicontinuous on* $B(u_0, \Delta) \cap (S_0, \alpha)$. *Then for any* $\varepsilon_n \to 0$, *there exists the sequence* u_n *of the local minimum solutions of* $P(\varepsilon_n)$ *such that* $u_n \to u_0$.

Proof. For any $\varepsilon_n \to 0$, by condition (a) there exists $n' \in N$ such that for $n \geqslant n'$, $U(\varepsilon_n) \subset U(U(0), \alpha)$, and $S_n \subset (S_0, \alpha)$. Without loss of generality we could suppose that $U(\varepsilon_n) \subset U(U(0), \alpha)$ and $S_n \subset (S_0, \alpha)$, $\forall n \in N$.

For any positive number ω: $0 < \omega < \min\{\alpha/2, \Delta\}$, by Lemmas 1 and 2 we have $S_0^1 \neq \varnothing$, $S_n^1 \neq \varnothing$, and the functional $J(0, u)$ satisfies inequality (4.7) for $u \in B(u_0, \omega) \cap (S_0, \alpha)$. By Lemma 5 there exists $n_1 \in N$ such that for $n \geqslant n_1$,

$$|J(\varepsilon_n, u) - J(0, u)| < \frac{\tau(\omega)}{4}, \qquad \forall u \in (S_0, \alpha). \tag{4.8}$$

Since (4.7), (4.8) hold for $\forall n \geqslant n_1$ and $\forall u \in S_n'$, we have

$$\begin{aligned} J(\varepsilon_n, u_0) &< J(0, u_0) + \frac{\tau(\omega)}{4} \leqslant J(0, u) - \tau(\omega) + \frac{\tau(\omega)}{4} \\ &< J(\varepsilon_n, u) - \tau(\omega) + \frac{\tau(\omega)}{4} + \frac{\tau(\omega)}{4} = J(\varepsilon_n, u) - \frac{\tau(\omega)}{2}; \end{aligned}$$

that is,

$$J(\varepsilon_n, u_0) < \inf\{J(\varepsilon_n, u) \mid u \in S_n^1\}. \tag{4.9}$$

Because u_0 may not belong to S_n and S_ε is open at $\varepsilon = 0$, there exists a sequence $(u_n \in S_n)_N$ such that u_n converges to u_0. By Lemma 5 we have $J(\varepsilon_n, u_n) \to J(0, u_0)$. Hence there exists $n_0 \in N$ such that for $n \geqslant n_0$,

$$J(\varepsilon_n, u_n) < \inf\{J(\varepsilon_n, u) \mid u \in S_n^1\}, \tag{4.10}$$

and $u_n \in S_n \cap \mathring{B}(u_0, \alpha)$ for $\forall n \geqslant n_0$, where $\mathring{B}(u_0, \omega)$ is the interior of $B(u_0, \omega)$.

Equation (4.10) provides that if $J(\varepsilon_n, u)$ have minimum solutions on $S_n \cap B(u_0, \omega)$, then this solution belongs to $\mathring{B}(u_0, \omega)$, because $B(u_0, \omega) \subset L_2(I)$ is a weakly compact set and S_n is a weakly closed set of $L_2(I)$, so that $B(u_0, \omega) \cap S_n$ is a weakly compact set. By condition (III), $J(\varepsilon_n, u)$ has a minimum solution $\bar{u}_n$ on $B(u_0, \omega) \cap S_n = S_n'$ and $\bar{u}_n \in \mathring{B}(u_0, \omega)$, $\forall n \geqslant n_0$. So there exists a real number sequence $(\omega_n)_N$, $\omega_n < \omega$ such that $B(\bar{u}_n, \omega_n) \subset \mathring{B}(u_0, \omega)$. Therefore, $\bar{u}_n$ is a local minimum solution of $P(\varepsilon_n)$. Since $\omega > 0$, owing to its arbitrariness (sufficiently small), obviously the sequence $(\bar{u}_n)_N$ may be selected to make $\bar{u}_n$ converge to u_0.

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